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SEPARATION OF VARIABLES FOR QUANTUM INTEGRABLE SYSTEMS ON ELLIPTIC CURVES

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Abstract. We extend Sklyanin's method of separation of variables to quantum integrable models associated to elliptic curves. After reviewing the differential case, the elliptic Gaudin model studied by Enriquez, Feigin and Rubtsov, we consider the difference case and find a class of transfer matrices whose eigenvalue problem can be solved by separation of variables. These transfer matrices are associated to representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ by difference operators. One model of statistical mechanics to which this method applies is the IRF model with antiperiodic boundary conditions. The eigenvalues of the transfer matrix are given as solutions of a system of quadratic equations in a space of higher order theta functions.

1. Introduction

The method of separation of variables in integrable lattice models, proposed by Sklyanin, is a method to find eigenvalues and eigenvectors of transfer matrices. It is an alternative to the Bethe ansatz and works in some situations where the Bethe ansatz does not, and gives an insight in the completeness of the Bethe ansatz. The method is closely related to Baxter's method (Chapter 9 of [2]), and in fact the eigenvalue problem in the separated variables (in the difference case) becomes the Baxter difference equation. In the Gaudin model, one of the simplest quantum integrable systems, this method relates the problem of finding common eigenvectors of Gaudin Hamiltonians to the problem of finding differential equations on the Riemann sphere with regular singular points whose monodromy is trivial. As noticed by Feigin and Frenkel (see [9]), this is a special case of the Beilinson-Drinfeld "geometric Langlands correspondance" relating certain local systems on a complex curve to $\mathcal{D}$-modules on moduli spaces of principal bundles on the curve.

Both for quantum integrable models and for the connection to the Langlands program, it may be interesting to extend the method of separation of variables to more general models.

The class of quantum integrable systems (families of commuting operators) one considers in this context arise in different classes. There are the differential models, such as the quantization of the Hitchin systems. They are given by families of commuting differential operators and are associated to complex curves. The Gaudin operators are the operators associated to the Riemann sphere. More generally one considers difference or q-deformed models, such as the six-vertex model, which degenerate to the differential

model when the parameter q tends to one. These models appear in three sorts: rational, trigonometric and elliptic, depending on the type of coefficients in the commuting operators.

From the point of view of representation theory, the differential models are related to Kac–Moody Lie algebras, and the q-deformed models to (infinite dimensional) quantum groups.

We consider here models related to sl_2. In the differential rational (genus zero) case the separation of variables was considered by Sklyanin [10] and Frenkel [9]. A version of the separation of variables for the genus one differential case was considered by Enriquez, Feigin and Rubtsov [4], who also made an explicit connection to the Langlands correspondance. The q-deformed rational and trigonometric case were studied by Sklyanin [10] and Tarasov–Varchenko [12]. The latter authors introduce the notion of a difference equation with regular singular points, thus giving a q-version of the relation described above for Gaudin models.

Here we consider the q-deformed elliptic case. The class of difference operator we give is both a q-deformation of the Enriquez–Feigin–Rubtsov differential operators and an elliptic version of the operators studied by Tarasov–Varchenko. Common eigenfuncions may be constructed by the method of separation of variables. Moreover the commuting difference operators we introduce can be restricted to functions on a finite subset of points. This restriction turns out to give the commuting transfer matrices of interaction-round-a-face models with antiperiodic boundary condition. This model provides an example of a model solvable by separation of variables but not by Bethe ansatz. Other elliptic models, related to the XYZ model, have recently been studied by the method of separation of variables by Sklyanin and Takebe [11].

The algebraic structure at the origin of our constructions is the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Indeed, the starting point is the construction of a representation of this quantum group by difference operators which generalizes the "universal evaluation module" of [7].

The paper is organized as follows:

In Section 2 we review the separation of variables in the differential elliptic case. Most of this part is essentially taken from [4], but we add some remarks on the Bethe ansatz and its completeness.

In Section 3 we explain what is needed from the theory of elliptic quantum groups and introduce a class of representations of $E_{\tau\eta}(sl_2)$ by difference operators and relate them to known representations. Twisted commuting transfer matrices are then introduced and the method of separation of variables is applied to construct (Bethe ansatz) eigenvectors.

In Section 4 we consider the restriction of the transfer matrix associated to the tensor product of n fundamental representations to functions of a finite set of cardinality 2^n, and show that we obtain the transfer matrix of an IRF model with antiperiodic boundary conditions. The eigenvalues are then obtained as the solutions of a system of n quadratic equations in an n-dimensional space of theta functions of order n.

In an appendix we give an account on “elliptic polynomials”, which are (twisted) theta functions of order n.

2. The differential case

Let us start by introducing a family of commuting differential operators associated to an elliptic curve with n marked points and n highest weight representations of $sl_2(\mathbb{C})$.

Let the elliptic curve be $E = \mathbb{C}/\Gamma$ with $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ and $\operatorname{Im}\tau > 0$. The marked points are the projections of n points $z_1, \dots, z_n \in \mathbb{C}$, with $z_i \neq z_j \mod \Gamma$ for $i \neq j$. The representations are $M_{\Lambda_1}, \dots, M_{\Lambda_n}$ where, for $\Lambda \in \mathbb{C}$, M_Λ denotes the Verma module (defined below in 2.1 of highest weight Λ of $sl_2(\mathbb{C})$.

Thus our parameters are $\tau, z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n$.

Let $\theta(z) = -\sum_{j\in\mathbb{Z}} \exp(i\pi(j+1/2)^2\tau + 2\pi i(j+1/2)(z+1/2))$ be the odd Jacobi theta function, and set $\sigma_\lambda(z) = \frac{\theta(\lambda-z)\theta'(0)}{\theta(z)\theta(\lambda)}$. It is the unique meromorphic function of z regular on $\mathbb{C}-\Gamma$, with a simple pole with residue one at 0, and such that $\sigma_\lambda(z+r+s\tau) = e^{2\pi i s\lambda}\sigma_\lambda(z)$, $r, s \in \mathbb{Z}$.

Let $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the standard generators of $sl_2(\mathbb{C})$. For $a \in sl_2(\mathbb{C})$ let $a^{(i)}$ denote the action of a on the ith factor of the tensor product $M = M_{\Lambda_1} \otimes \cdots \otimes M_{\Lambda_n}$.

Introduce the following endomorphisms of M depending on $z, \lambda \in \mathbb{C}$:

$$h(z) = \sum_{i=1}^{n} \frac{\theta'(z-z_i)}{\theta(z-z_i)} h^{(i)}, \qquad e_\lambda(z) = \sum_{i=1}^{n} \sigma_{-\lambda}(z-z_i) e^{(i)}, \qquad f_\lambda(z) = \sum_{i=1}^{n} \sigma_\lambda(z-z_i) f^{(i)}.$$

The family of commuting differential operators is then obtained by the following generating function, which is an elliptic version of the generating function of Gaudin Hamiltonians [10].

Theorem 2.1. *Let for $z \in \mathbb{C}$, $S(z)$ be the differential operator acting on functions of one complex variables λ with values in the zero weight space $M[0] = \{v \in M, \sum_i h^{(i)} v = 0\}$ of M.*

$$S(z) = \left(\frac{\partial}{\partial\lambda} - \frac{1}{2}h(z)\right)^2 + e_\lambda(z) f_\lambda(z) + f_\lambda(z) e_\lambda(z).$$

Then $S(z)S(w) = S(w)S(z)$.

One way of proving this theorem is to notice that it is a special case of the flatness of the KZB connection (Prop. 2 in [5]). The relation to the KZB connection is the following. In the $sl_2(\mathbb{C})$ case, the KZB connection involves the differential operators (appearing on the right hand side of the KZB equations)

$$H_j = -h^{(j)}\frac{\partial}{\partial\lambda} + \sum_{k:k\neq j} \frac{1}{2}\frac{\theta'(z_j-z_k)}{\theta(z_j-z_k)} h^{(j)}h^{(k)} + \sigma_\lambda(z_j-z_k)e^{(j)}f^{(k)} + \sigma_{-\lambda}(z_j-z_k)f^{(j)}e^{(k)},$$

$j = 1, \dots, n$ and

$$H_0 = \frac{\partial^2}{\partial\lambda^2} + \frac{1}{2}\sum_{j,k=1}^{n} \frac{1}{2} h^{(j)} h^{(k)} \frac{\theta''(z_j - z_k)}{\theta(z_j - z_k)} - (e^{(j)} f^{(k)} + f^{(j)} e^{(k)}) \frac{\partial\sigma_\lambda(z_j - z_k)}{\partial\lambda}.$$

The fact that the connection is flat means in particular that these operators form a commuting family when acting on $M[0]$-valued functions. The terms with $j = k$ are understood as the limit as the argument $z_j - z_k$ tends to zero. So $\theta''(0)/\theta(0)$ means $\theta'''(0)/\theta'(0)$ and $(\partial\sigma_\lambda/\partial\lambda)(0) = (\theta'/\theta)'(\lambda)$. Note that $\sum_{j=1}^n H_j$ vanishes on $M[0]$-valued functions. We call the operators H_j, $j \geq 1$, the elliptic Gaudin Hamiltonian and H_0 the (generalized) Lamé Hamiltonian (it is the Lamé operator if $n = 1$).

The relation between these commuting operators and $S(z)$ is:

Proposition 2.2. *Let*[1] $\bar\zeta(z) = \theta'(z)/\theta(z)$, $\bar\wp(z) = -\bar\zeta'(z)$. *Then*

$$S(z) = \sum_{k=1}^{n} \frac{c_k}{2} \bar\wp(z - z_k) + \sum_{k=1}^{n} H_k \bar\zeta(z - z_k) + H_0,$$

and c_j *is the Casimir value* $c_j = \frac{1}{2}\Lambda_j(\Lambda_j + 2)$.

The proof of this fact follows by noting that $S(z)$ is a meromorphic doubly periodic function of z with at most double poles at the points z_j. By expanding $S(z)$ in a Laurent series up to the constant term at the $z = z_j$, we find that the difference between left-hand side and right-hand side is a differential operator whose coefficients are regular elliptic functions vanishing at least at one point. Such an operator vanishes by Liouville's theorem.

The eigenvalue problem for common eigenfunctions of $H_0, \dots, H_n$ can then be formulated as $S(z)u = q(z)u$ with $q(z) = \sum_{k=1}^n \frac{c_k}{2}\bar\wp(z - z_k) + \sum_{k=1}^n \epsilon_k \bar\zeta(z - z_k) + \epsilon_0$. The eigenvalue of H_j is then ϵ_j, $j = 0, \dots, n$. Since $\sum_{j\geq 1} H_j = 0$, one must necessarily have $\epsilon_1 + \cdots + \epsilon_n = 0$.

Common eigenfunctions of H_j and thus of $S(z)$ can be obtained by the Bethe ansatz method. They have the form $f(w_1)\cdots f(w_m)v_0$ where v_0 is the tensor product of highest weight vectors, $m = \frac{1}{2}\sum \Lambda_j$ and $w_1, \dots, w_m$ are a solution to the system of Bethe ansatz equations, see [5, 6] and below.

2.1. **Separation of variables.** We realize the representations of $sl_2(\mathbb{C})$ by differential operators:

Lemma 2.3. *For any* $\Lambda \in \mathbb{C}$, *the map* $f \mapsto t$, $h \mapsto -2t\frac{d}{dt} + \Lambda$, $e \mapsto -t\frac{d^2}{dt^2} + \Lambda\frac{d}{dt}$ *defines a representation of* $sl_2(\mathbb{C})$ *on* $\mathbb{C}[t]$, *the Verma module* M_Λ. *If* Λ *is a nonnegative integer,* $t^{\Lambda+1}\mathbb{C}[t]$ *is an invariant subspace and the quotient* $L_\Lambda = \mathbb{C}[t]/t^{\Lambda+1}\mathbb{C}[t]$ *is irreducible with highest weight vector* $1 \in \mathrm{Ker}(e)$ *of weight* Λ.

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[1] The relation of these functions with the classical Weierstrass ζ and $\wp$ functions $\zeta(z) = \frac{1}{z} + \sum_{(r,s)\in\mathbb{Z}^2-(0,0)} \frac{1}{z+r+s\tau} - \frac{1}{r+s\tau} + \frac{z}{(r+s\tau)^2}$, $\wp(z) = -\zeta'(z)$ is $\zeta(z) = \bar\zeta(z) + 2\eta_1 z$, $\wp(z) = \bar\wp(z) - 2\eta_1$, where $2\eta_1 = \theta'''(0)/3\theta'(0)$.

The proof consists of checking the relations $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$ and $e\,t^{\Lambda+1}=0$.

Therefore we may realize the tensor product M as $\mathbb{C}[t_1,\dots,t_n]$, and the tensor product of irreducible representations (for integer Λ_j) as $\mathbb{C}[t_1,\dots,t_n]/\sum_j(t_j^{\Lambda_j+1}\mathbb{C}[t_1,\dots,t_n])$. Then $M[0]$ consists of homogeneous polynomials of degree $m=\sum\Lambda_k/2$. We may then view $e_\lambda(z), f_\lambda(z), h(z)$, $S(z)$, H_j as differential operators in $n+1$ variables $\lambda, t_1,\dots,t_n$. They commute when acting on functions which are homogeneous in $t_1,\dots,t_n$ of degree m.

2.2. **Separated variables.** We express the differential operators $S(z)$ in terms of new variables so that the eigenvalue problem is reduced to ordinary differential equations. Following Sklyanin's idea, the new variables $C, y_1,\dots,y_n$ are the zeros and the leading coefficient of the operator f_λ:

$$f_\lambda(z)=C\prod_{j=1}^{n}\frac{\theta(z-y_j)}{\theta(z-z_j)}$$

Since $f_\lambda(z)$ is realized as a multiplication operator and both sides of this equation are functions of z with definite transformation properties under translations by the lattice Γ, this equation does define, locally around a generic point, a biholomorphic change of variables $(C,y_1,\dots,y_n)\mapsto(\lambda,t_1,\dots,t_n)$. The formulae are

$$t_i=C\frac{\prod_j\theta(z_i-y_j)}{\theta'(0)\prod_{j:j\neq i}\theta(z_i-z_j)},$$

and

$$\lambda=\sum_{j=1}^{n}(y_j-z_j).$$

From these formulae we deduce the transformation properties of partial derivatives:

$$\frac{\partial}{\partial y_j}=\frac{\partial}{\partial\lambda}+\sum_{k=1}^{n}\frac{\theta'(y_j-z_k)}{\theta(y_j-z_k)}\,t_k\frac{\partial}{\partial t_k}\,.$$

$$C\frac{\partial}{\partial C}=\sum_{k=1}^{n}t_k\frac{\partial}{\partial t_k}.$$

The next step is to remark that a function $u(C,y_1,\dots,y_n)$ obeys $S(z)u=q(z)u$ with $q(z)=\sum_{k=1}^n\frac{c_k}{2}\bar{\wp}(z-z_k)+\sum_{k=1}^n\epsilon_k\bar{\zeta}(z-z_k)+\epsilon_0$ and $\sum_{j\geq1}\epsilon_j=0$ if and only if it obeys $S(y_j)u=q(y_j)u$ for all $j=1,\dots,n$ and all generic points $(y_1,\dots,y_n)$. Here the ambiguous notation $S(y_j)$ means: write differential operator $S(z)$ with the coefficients on the left of the partial derivatives and replace z by y_j in the coefficients. To prove this statement notice that, if $S(y_j)v=q(y_j)v$, then $\prod\theta(z-z_i)(S(z)-q(z))u(y_1,\dots,y_n)$ is

a holomorphic theta function in z of order n vanishing at n generic points y_i. It thus vanishes, see the Appendix.

It is then convenient to use the identity $[e_\lambda(z), f_\lambda(z)] = -h'(z)$ on $M[0]$, to write $S(z)$ as $S(z) = \left(\frac{\partial}{\partial\lambda} - \frac{1}{2}h(z)\right)^2 - h'(z) + 2f_\lambda(z)e_\lambda(z)$. so that the last term vanishes if we set $z = y_j$ and we get

$$S(y_j) = \left(\frac{\partial}{\partial y_j} - \sum_{k=1}^{n} \frac{\Lambda_k}{2}\bar{\zeta}(y_j - z_k)\right)^2. \tag{1}$$

Proposition 2.4. *A function $u(\lambda, t_1, \ldots, t_n)$, homogeneous of degree $m = \frac{1}{2}\sum \Lambda_k$ in the t_i, is a local solution of the partial differential equations $S(z)u = q(z)u$, $z \in \mathbb{C}$ if and only if*

$$u(\lambda, t_1, \ldots, t_n) = C^n v(y_1, \ldots, y_n)$$

and v obeys

$$\nabla_{y_j}^2 v = q(y_j)v, \qquad \nabla_{y_j} = \frac{\partial}{\partial y_j} - \sum_{k=1}^{n} \frac{\Lambda_k}{2}\bar{\zeta}(y_j - z_k). \tag{2}$$

2.3. **Interpolation formula.** The formula (1) expresses the values at $y_1, \ldots, y_n$ of the coefficients of $S(z)$ for $z = y_j$. Since the coefficients of $S(z) - \frac{1}{2}\sum_k c_k \bar{\wp}(z - z_k)$ are elliptic functions of z with at most simple poles at $z_1, \ldots, z_n$, they are uniquely determined by these values, and can be calculated by an interpolation formula: let us write $S(z)$ in the form

$$S(z) = \frac{1}{2}\sum_{k=1}^{n} c_k \bar{\wp}(z - z_k) + \prod_{k=1}^{n} \theta(z - z_k)^{-1} \hat{S}(z),$$

so that $\hat{S}(z)$ is a theta function of order n. Thus (see the Appendix)

$$\hat{S}(z) = \sum_{i=1}^{n} \frac{\theta(z + \sum_{j\neq i} y_j - \sum_k z_k)}{\theta(\sum_j y_j - \sum_k z_k)} \prod_{j:j\neq i} \frac{\theta(z - y_j)}{\theta(y_i - y_j)} \hat{S}_i,$$

with

$$\hat{S}_i = \prod_{k=1}^{n} \theta(y_i - z_k) \left(\left(\frac{\partial}{\partial y_i} - \sum_{k=1}^{n} \frac{\Lambda_k}{2}\bar{\zeta}(y_i - z_k) \right)^2 - \frac{1}{2}\sum_{k=1}^{n} c_k \bar{\wp}(y_i - z_k) \right).$$

2.4. **Bethe ansatz.** The separated equation reads

$$\nabla_y^2 v - (\sum_{k=1}^{n} \frac{c_k}{2}\bar{\wp}(y - z_k) + \sum_{k=1}^{n} \epsilon_j \bar{\zeta}(y - z_k) + \epsilon_0)v = 0, \tag{3}$$

with $\nabla_y = \frac{\partial}{\partial y} - \sum_{k=1}^n \frac{\Lambda_k}{2}\bar\zeta(y-z_k)$. It is a second order ordinary differential equation with regular singular points at z_k and characteristic exponents 0 and Λ_k+1. Following Hermite's method to solve the Lamé equation (see [13]) we seek solutions of the form

$$v(y) = e^{cy}\prod_{k=1}^m \theta(y-w_k). \tag{4}$$

Functions of this form are called elliptic polynomials, see the Appendix.

Let us first assume that $w_k \neq z_j \mod \Gamma$ for all j,k. Then we also have $w_k \neq w_l$ for $k \neq l$, since the only solution vanishing with its derivative at a regular point is the trivial solution. Rewrite the equation in the form $v''(y) - \sum_k \Lambda_k\bar\zeta(y-z_k)v'(y) + b(y)v(y) = 0$ so that $b(y)$ has at most simple poles at the z_j. Taking derivatives of a function v of the form (4) and setting y equal to one of its zeros, we find the relation:

$$v''(w_k) = \sum_{j\neq k}\bar\zeta(w_k-w_j)v'(w_k) + 2cv'(w_k).$$

Inserting this into the differential equation, we see that v is a solution if and only if its zeros w_j obey the "Bethe ansatz equations"

$$\sum_{l=1}^n \Lambda_l\bar\zeta(w_j-z_l) - \sum_{k:k\neq j}\bar\zeta(w_j-w_k) = 2c, \qquad j=1,\dots,m.$$

Let us now consider the more general case of elliptic polynomials (4) vanishing at z_i, for i in some subset I of $\{1,\dots,n\}$. Since the characteristic exponents at z_i are $0, \Lambda_i+1$, a solution vanishing at z_i must vanish to order Λ_i+1 and is thus divisible by $\theta(y-z_i)^{\Lambda_i+1}$ (this is only possible if $\Lambda_i \in \mathbb{Z}_{\geq 0}$). Then $\tilde v(y) = \prod_{i\in I}\theta(y-z_i)^{-\Lambda_i-1}v(y)$ is again of the form (4), but with m replaced by $\tilde m = m - \sum_{i\in I}(\Lambda_i+1)$. It obeys the equation (3) with Λ_i replaced by $-\Lambda_i-2$ for $i\in I$.

Thus all elliptic polynomial solutions of (3) are of the form

$$v(y) = e^{cy}\prod_{i\in I}\theta(y-z_i)^{\tilde\Lambda_i+1}\prod_{k=1}^{\tilde m}\theta(y-w_k),$$

for some subset I of $\{j \,|\, \Lambda_j \in \mathbb{Z}_{\geq 0}\}$, such that $w_j \neq w_l \neq z_i$ $(j\neq l)$ and $w_1,\dots,w_{\tilde m}, c$ obey the Bethe ansatz equations

$$\sum_{l=1}^n \tilde\Lambda_l\bar\zeta(w_j-z_l) - \sum_{k:k\neq j}\bar\zeta(w_j-w_k) = 2c, \qquad j=1,\dots,\tilde m. \tag{5}$$

Here $\tilde\Lambda_l = -\Lambda_l - 2$ if $l\in I$ and $\tilde\Lambda_l = \Lambda_l$ otherwise.

To each such solution there corresponds a common eigenfunction u which, expressed in the separated variables, is $u = C^n\prod v(y_i)$. Up to a nonzero constant we get

$$u(\lambda) = e^{c\lambda}f(w_1)\cdots f(w_{\tilde m})v_I. \tag{6}$$

Here $v_I = \prod_{i\in I}(f^{(i)})^{\Lambda_i+1}v_0$ is a product of singular vectors. Only eigenvectors corresponding to $I=\emptyset$ have non-trivial projections to eigenvectors with values in the tensor product of irreducible representations.

Eigenvectors of the form (6), such that $w_1,\dots,w_{\tilde m},c$ are a solution of the Bethe ansatz equations (5) with $w_j\neq w_k \mod \Gamma$, $(j\neq k)$ and $w_j\neq z_i \mod\Gamma$ are called *Bethe eigenvectors*.

2.5. **Completeness of Bethe eigenvectors.** Let us consider the common eigenvalue problem

$$H_i u(\lambda)=\epsilon_i u(\lambda),\qquad i=0,\dots,n, \tag{7}$$

A natural class of functions preserved by the operators H_i is given by meromorphic sections of a flat line bundle on E. Namely, let for a character $\chi:\Gamma\to\mathbb{C}^\times$, $\mathcal{H}(\chi)$ be the space of meromorphic functions $\lambda\mapsto u(\lambda)\in M[0]$ such that $u(\lambda+1)=\chi(1)u(\lambda)$ and

$$u(\lambda+\tau)=\chi(\tau)e^{\pi i\sum_j z_jh^{(j)}}u(\lambda).$$

It is easy to see that $e_\lambda(z)$, $f_\lambda(z)$ and $\partial/\partial\lambda - h(z)/2$ preserve functions with these transformation properties, so that $S(z)$ and H_j preserve $\mathcal{H}(\chi)$.

It is then natural to look for eigenfunctions (non-trivial solutions of the differential equations (7)) in $\mathcal{H}(\chi)$.

Let $\Sigma(\chi)$ be the set of $(\epsilon_0,\dots,\epsilon_n)\in\mathbb{C}^{n+1}$ such that there exists a non-trivial function $u\in\mathcal{H}(\chi)$ with $H_ju=\epsilon_ju$, $j=0,\dots,n$. Let $\sigma\in\Gamma^*$ be the character such that $\sigma(r+s\tau)=(-1)^{r+s}$.

Theorem 2.5. *Let $\chi\in\Gamma^*$. Then $(\epsilon_0,\dots,\epsilon_n)\in\Sigma(\chi)$ if and only if $\sum_{k\geq1}\epsilon_k=0$ and the separated problem*

$$\nabla_y^2 v(y)-\frac14\sum_k\Lambda_k(\Lambda_k+2)\bar\wp(y-z_k)v(y)=(\sum_k\bar\zeta(y-z_k)\epsilon_k+\epsilon_0)v(y),$$

admits a non-trivial elliptic polynomial solution $v\in\Theta_m(\sigma^m\chi)$. In this case there is a Bethe eigenvector with these eigenvalues.

Proof: A common eigenfunction, viewed as a polynomial in t_i has the form

$$u(\lambda,t_1,\dots,t_n)=\sum_{m_1+\cdots+m_n=m}u_{m_1,\dots,m_n}(\lambda)\prod_i t_i^{m_i}.$$

Replacing t_i,λ as functions of the new variables, we obtain

$$u(\lambda,t_1,\dots,t_n)=C^n v(y_1,\dots,y_n),$$

where $v(y_1,\dots,y_n)$ is a linear combination of products of theta functions in each of the y_j with coefficients $u_{m_1,\dots,m_n}(\sum y_j-\sum z_k)$ and $v(y_1,\dots,y_n)$ obeys, in each variable, the separated second order equation (2). A priori the meromorphic function $v(y_1,\dots,y_n)$ may have poles on the hyperplane $\sum y_i=\sum z_i \mod \Gamma$. However this is impossible: consider v as a function of, say y_1 with the other variable fixed at some generic position.

Then v, as a function of y_1, being a solution of a linear second order equation may only have singularities at the poles z_j of the coefficients. Moreover, since u is in $H(\chi)$, the functions $y_i \mapsto v(y_1,\dots,y_n)$ belong to $\Theta_m(\sigma^m\chi)$. Thus the separated problem admits a non-trivial solution in $\Theta_m(\sigma^m\chi)$. As shown in the previous subsection, such a solution gives rise to a Bethe eigenvector. $\square$

3. The difference case

3.1. **Representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$.** The difference version of the differential operators $e_\lambda(z), f_\lambda(z), \partial_\lambda - h(z)/2$, are operators obeying the relations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Let us recall the definitions [7]: we fix two complex parameters τ,η, such that $\mathrm{Im}(\tau)>0$. The definition of $E_{\tau,\eta}(sl_2)$ is based on a dynamical R-matrix $R(z,\lambda)$ which we now introduce. Let

$$\alpha(z,\lambda)=\frac{\theta(\lambda+2\eta)\theta(z)}{\theta(\lambda)\theta(z-2\eta)},\qquad \beta(z,\lambda)=-\frac{\theta(\lambda+z)\theta(2\eta)}{\theta(\lambda)\theta(z-2\eta)},$$

Let V be a two dimensional complex vector space with basis $e[1], e[-1]$, and let $E_{ij}e[k]=\delta_{jk}e[i]$, $h=E_{11}-E_{-1,-1}$. Then, for $z,\lambda\in\mathbb{C}$, $R(z,\lambda)\in\mathrm{End}(V\otimes V)$ is the matrix

$$\begin{aligned}R(z,\lambda) &= E_{11}\otimes E_{11}+E_{-1,-1}\otimes E_{-1,-1}+\alpha(z,\lambda)E_{11}\otimes E_{-1,-1}\\ &+ \alpha(z,-\lambda)E_{-1,-1}\otimes E_{11}+\beta(z,\lambda)E_{1,-1}\otimes E_{-1,1}+\beta(z,-\lambda)E_{-1,1}\otimes E_{1,-1}.\end{aligned}$$

This R-matrix obeys the dynamical quantum Yang–Baxter equation

$$\begin{aligned}&R^{(12)}(z-w,\lambda-2\eta h^{(3)})R^{(13)}(z,\lambda)\,R^{(23)}(w,\lambda-2\eta h^{(1)})=\\ &\qquad R^{(23)}(w,\lambda)\,R^{(13)}(z,\lambda-2\eta h^{(2)})R^{(12)}(z-w,\lambda)\end{aligned}$$

in $\mathrm{End}(V\otimes V\otimes V)$, $z,w,\lambda\in\mathbb{C}$. The meaning of this notation is the following: $R^{(12)}(\lambda-2\eta h^{(3)})\,v_1\otimes v_2\otimes v_3$ is defined as

$$(R(z,\lambda-2\eta\mu_3)\,v_1\otimes v_2)\otimes v_3,$$

if $hv_3=\mu_3v_3$. The other terms are defined similarly: in general, let $V_1,\dots,V_n$ be modules over the one dimensional Lie algebra $\mathfrak{h}=\mathbb{C}h$ with one generator h, such that, for all i, V_i is the direct sum of finite dimensional eigenspaces $V_i[\mu]$ of h, labeled by the eigenvalue μ. We call such modules diagonalizable $\mathfrak{h}$-modules. If $X\in\mathrm{End}(V_i)$ we denote by $X^{(i)}\in\mathrm{End}(V_1\otimes\cdots\otimes V_n)$ the operator $\cdots\otimes\mathrm{Id}\otimes X\otimes\mathrm{Id}\otimes\cdots$ acting non-trivially on the ith factor, and if $X=\sum X_k\otimes Y_k\in\mathrm{End}(V_i\otimes V_j)$ we set $X^{(ij)}=\sum X_k^{(i)}Y_k^{(j)}$. If $X(\mu_1,\dots,\mu_n)$ is a function with values in $\mathrm{End}(V_1\otimes\cdots\otimes V_n)$, then $X(h^{(1)},\dots,h^{(n)})v=X(\mu_1,\dots,\mu_n)v$ if $h^{(i)}v=\mu_iv$, for all $i=1,\dots,n$.

A *module over* $E_{\tau,\eta}(sl_2)$ is then a diagonalizable $\mathfrak{h}$-module $W=\oplus_{\mu\in\mathbb{C}}W[\mu]$, together with an L-operator $L(z,\lambda)\in\mathrm{End}_{\mathfrak{h}}(V\otimes W))$ (a linear map commuting with $h^{(1)}+h^{(2)}$) depending meromorphically on $z,\lambda\in\mathbb{C}$ and obeying the relations

$$\begin{aligned}&R^{(12)}(z-w,\lambda-2\eta h^{(3)})L^{(13)}(z,\lambda)\,L^{(23)}(w,\lambda-2\eta h^{(1)})=\\ &\qquad L^{(23)}(w,\lambda)\,L^{(13)}(z,\lambda-2\eta h^{(2)})\,R^{(12)}(z-w,\lambda)\,.\end{aligned}\tag{8}$$

For example, $W = V$, $L(w,\lambda) = R(w - z_0, \lambda)$ is a module over $E_{\tau,\eta}(sl_2)$, called the fundamental representation, with evaluation point z_0. In [7] more general examples of such modules were constructed: in particular, for any pair of complex numbers Λ, z we have an *evaluation Verma module* $M_\Lambda(z)$. It has a weight decomposition $M_\Lambda = \oplus_{j=0}^{\infty} M_\Lambda[\Lambda - 2j]$, with one-dimensional weight spaces $M_\Lambda[\mu]$. The action of the L-operator is described explicitly in [7]. Also, we have a notion of tensor products of modules over $E_{\tau,\eta}(sl_2)$. The main examples considered in this paper will be tensor products $M_{\Lambda_1}(z_1) \otimes \cdots \otimes M_{\Lambda_n}(z_n)$ of evaluation Verma modules and some of their subquotients.

It will be convenient here to consider more general L-operators obeying the relations. So we define a *functional module* over $E_{\tau,\eta}(sl_2)$ to be given by a pair (W, L) where W is a space of complex-valued functions on a certain set and h acts on it as multiplication by a function, and $L(z,\lambda)$ is a meromorphic function of z and λ acting as a difference operator on $V \otimes W$, commuting with $h \otimes 1 + 1 \otimes h$, and obeying the relations (8). An example of such functional modules is provided by the "universal evaluation modules" of [7]. h acts by multiplication by a continuous variable. Evaluation Verma modules are obtained by restricting the range of this continuous variable to a discrete set.

For any module or functional module W over $E_{\tau,\eta}(sl_2)$, we define the associated operator algebra, an algebra of operators on the space $\mathrm{Fun}(W)$ of meromorphic functions of $\lambda \in \mathbb{C}$ with values in W. It is generated by h, acting on the values, and operators $a(z), b(z), c(z), d(z)$. Namely, let $\tilde{L}(z) \in \mathrm{End}(V \otimes \mathrm{Fun}(W))$ be the operator defined by $(\tilde{L}(z)(v \otimes f))(\lambda) = L(z,\lambda)(v \otimes f(\lambda - 2\eta\mu))$ if $hv = \mu v$. View $\tilde{L}(z)$ as a 2 by 2 matrix with entries in $\mathrm{End}(\mathrm{Fun}(W))$:

$$\begin{aligned} \tilde{L}(z)(e[1] \otimes f) &= e[1] \otimes a(z)f + e[-1] \otimes c(z)f, \\ \tilde{L}(z)(e[-1] \otimes f) &= e[1] \otimes b(z)f + e[-1] \otimes d(z)f. \end{aligned}$$

The relations obeyed by these operators are described in detail in [7] (in [7] these operators are denoted by $\tilde{a}(z), \tilde{b}(z)$ and so on).

To each module we associate a central element of the operator algebra. It is given by the quantum determinant [7]

$$\mathrm{Det}(z) = \frac{\theta(\lambda)}{\theta(\lambda - 2\eta h)}(a(z + 2\eta)d(z) - c(z + 2\eta)b(z)).$$

3.2. **A class of representations by difference operators.** Let $z_1, \ldots, z_n \in \mathbb{C}$ be distinct points and $\Lambda_1, \ldots, \Lambda_n \in Z_{\geq 0}$. Let us introduce difference operators acting on functions of $n+1$ complex variables $\lambda, x_1, \ldots, x_n$. Let $(T^a_{x_i} f)(\lambda, x_1, \ldots, x_n) = f(\lambda, x_1, \ldots, x_i + a, \ldots, x_n)$ and $(T^a_\lambda f)(\lambda, x_1, \ldots, x_n) = f(\lambda + a, x_1, \ldots, x_n)$. The steps a will always be $\pm 2\eta$. Let

$$\Delta_-(z) = \prod_{i=1}^{n} \theta(z - z_i + \Lambda_i \eta) \text{ and } \Delta_+(z) = \prod_{i=1}^{n} \theta(z - z_i - \Lambda_i \eta)\,.$$

The functions $\Delta_\pm(z=-x_i)$, considered as multiplication operators, will be denoted simply by $\Delta_\pm(-x_i)$. We also set $s=\sum_{i=1}^n(x_i+z_i)$.

With these conventions, we define:

$$
\begin{aligned}
a(z) &= \prod_{i=1}^n \theta(z+x_i)\frac{\theta(\lambda+\sum_{l=1}^n(x_l+z_l+\Lambda_l\eta))}{\theta(\lambda)}T_\lambda^{-2\eta} \\
b(z) &= -\sum_{i=1}^n \frac{\theta(\lambda+z+x_i)}{\theta(\lambda)}\prod_{j\neq i}\frac{\theta(z+x_j)}{\theta(x_i-x_j)}\Delta_+(-x_i)T_{x_i}^{-2\eta}T_\lambda^{+2\eta}, \\
c(z) &= -\sum_{i=1}^n \frac{\theta(-\lambda+z+x_i-2s)}{\theta(\lambda)}\prod_{j\neq i}\frac{\theta(z+x_j)}{\theta(x_i-x_j)}\Delta_-(-x_i)T_{x_i}^{+2\eta}T_\lambda^{-2\eta}, \\
\mathrm{Det}(z) &= \prod_{i=1}^n \theta(z-z_i-\Lambda_i\eta)\,\theta(z-z_i+\Lambda_i\eta+2\eta)
\end{aligned}
$$

Theorem 3.1. *The difference operators $a(z),b(z),c(z)$ together with $d(z)$ defined implicitly by the determinant relation*

$$
a(z+2\eta)d(z)-c(z+2\eta)b(z)=\frac{\theta(\lambda-2\eta h)}{\theta(\lambda)}\mathrm{Det}(z)
$$

obey the relations of the elliptic quantum group (8) with $\eta h=-\sum_i(x_i+z_i)$.

Example. If $n=1$, the generators act on functions of two variables x_1,λ, and h acts as $-x_1-z_1$. If we introduce a new variable $h=-\eta^{-1}(x_1+z_1)$, so that the generator h acts by multiplication by h, we obtain a representation on functions of h,λ. the action of the generators is given by the difference operators

$$
\begin{aligned}
a(z)v(h,\lambda) &= \theta(z-z_1-\eta h)\frac{\theta(\lambda-\eta h+\Lambda_1\eta)}{\theta(\lambda)}v(h,\lambda-2\eta) \\
b(z)v(h,\lambda) &= \frac{\theta(\lambda+z-z_1-\eta h)}{\theta(\lambda)}\theta(-\eta h+\Lambda_1\eta)v(h+2,\lambda+2\eta), \\
c(z)v(h,\lambda) &= -\frac{\theta(-\lambda+z-z_1+\eta h)}{\theta(\lambda)}\theta(\eta h+\Lambda_1\eta)v(h-2,\lambda-2\eta), \\
d(z)v(h,\lambda) &= \theta(z-z_1+\eta h)\frac{\theta(\lambda-\eta h-\Lambda_1\eta)}{\theta(\lambda)}v(h,\lambda+2\eta).
\end{aligned}
$$

This is, up to normalization, the "universal evaluation module" of [7], Sect. 9.

Proof of Theorem 3.1: The proof consists of a straightforward verification of the sixteen relations. Let us give an example: one relation is

$$
a(z)b(w)=\frac{\theta(z-w)\theta(\lambda+2\eta)}{\theta(z-w-2\eta)\theta(\lambda)}b(w)a(z)+\frac{\theta(z-w-\lambda)\theta(2\eta)}{\theta(z-w-2\eta)\theta(\lambda)}a(w)b(z)\,.
$$

This identity is verified by looking at each summand of $b(z)$ and $b(w)$ separately. The corresponding equation to one such summand typically looks like

$$\theta(z+x_k)(\prod_{i\neq k}\theta(z+x_i))\frac{\theta(\lambda+\sum_{i=1}^n(x_i+z_i+\Lambda_i\eta))\theta(\lambda+w+x_k-2\eta)}{\theta(\lambda)\theta(\lambda-2\eta)}\times$$

$$\times(\prod_{i\neq k}\frac{\theta(w+x_i)}{\theta(x_k-x_i)})\Delta_+(-x_k)T_{x_k}^{-2\eta}$$

$$= \frac{\theta(z-w)\theta(\lambda+2\eta)}{\theta(z-w-2\eta)\theta(\lambda)}\frac{\theta(\lambda+w+x_k)}{\theta(\lambda)}(\prod_{i\neq k}\frac{\theta(w+x_i)}{\theta(x_k-x_i)})\times$$

$$\times(\prod_{i\neq k}\theta(z+x_i))\frac{\theta(z+x_k-2\eta)\theta(\lambda+\sum_{i=1}^n(x_i+z_i+\Lambda_i\eta))}{\theta(\lambda+2\eta)}\Delta_+(-x_k)T_{x_k}^{-2\eta}$$

$$+ \frac{\theta(z-w-\lambda)\theta(2\eta)}{\theta(z-w-2\eta)\theta(\lambda)}\frac{\theta(\lambda+\sum_{i=1}^n(x_i+z_i+\Lambda_i\eta))\theta(w+x_k)}{\theta(\lambda)}(\prod_{i\neq k}\theta(w+x_i))\times$$

$$\times(\prod_{i\neq k}\frac{\theta(z+x_i)}{\theta(x_k-x_i)})\frac{\theta(z+x_k+\lambda-2\eta)}{\theta(\lambda-2\eta)}\Delta_+(-x_k)T_{x_k}^{-2\eta}\,.$$

By taking into account that each summand of the above equation involves a factor

$$(\prod_{i\neq k}\frac{\theta(z+x_i)\theta(w+x_i)}{\theta(x_k-x_i)})\frac{\theta(\lambda+\sum_{i=1}^n(x_i+z_i+\Lambda_i\eta))}{\theta(\lambda)}\,,$$

the task reduces to verifying

$$\theta(z+x_k)\frac{\theta(\lambda+w+x_k-2\eta)}{\theta(\lambda-2\eta)} = \frac{\theta(z-w)\theta(\lambda+w+x_k)\theta(z+x_k-2\eta)}{\theta(z-w-2\eta)\theta(\lambda)}$$

$$+\frac{\theta(z-w-\lambda)\theta(2\eta)\theta(w+x_k)\theta(\lambda+z+x_k-2\eta)}{\theta(z-w-2\eta)\theta(\lambda)\theta(\lambda-2\eta)}\,,$$

which we write in the form

$$f_1(z,w,\lambda)=f_2(z,w,\lambda)+f_3(z,w,\lambda)\,.$$

This identity is proved in two steps. First, one shows that the functions $f_i(z,w,\lambda)$, $i=1,2,3$, transform in the same way under $\lambda\to\lambda+1$, $\lambda\to\lambda+\tau$,. The transformation laws thus obtained are the following:

$$f_i(z,w,\lambda+1)=f_i(z,w,\lambda)$$
$$f_i(z,w,\lambda+\tau)=e^{-2\pi i(w+x_k)}f_i(z,w,\lambda).$$

for $i=1,2,3$. The second step is to show that the above relation holds for the residues of the functions $f_i(z,w,\lambda)$. Here, one has to show that the identity holds for $\lambda=2\eta$

and $\lambda = 0$. For $\lambda = 0$ one obtains

$$\frac{\theta(z-w)\theta(w+x_k)\theta(z+x_k-2\eta)}{\theta(z-w-2\eta)} + \frac{\theta(z-w)\theta(2\eta)\theta(w+x_k)\theta(z+x_k-2\eta)}{\theta(-2\eta)\theta(z-w-2\eta)} = 0\,,$$

whereas the $\lambda = 2\eta$ residue yields

$$\theta(z+x_k)\theta(w+x_k) = \frac{\theta(z-w-2\eta)\theta(2\eta)}{\theta(z-w-2\eta)\theta(2\eta)}\theta(w+x_k)\theta(z+x_k)\,.$$

This proves [7], paragraph 3, relation 2. The other relations are proved in a similar but often more intricate fashion.

One identity that is used is the vanishing of the sum of the residues at $v = -x_j - 2\eta$, $-x_j$ of the function

$$f(v) = \frac{\theta(2s+x_i+v+2\eta)}{\theta(v+x_i+2\eta)} \prod_{l=1}^{n} \frac{\theta(v-z_l-\Lambda_l)\theta(v-z_l+\Lambda_l+2\eta)}{\theta(v+x_l)\theta(v+x_l+2\eta)}\,,$$

a consequence of the double periodicity $f(v+1) = f(v+\tau) = f(v)$.

Also some of the more complicated relations, such as $d(z)d(w) = d(w)d(z)$ turn out to be consequences of the simpler relations and the fact that the determinant is central. □

Remark. This is the difference elliptic analogue of formulae that have appeared in the literature. In the rational difference and differential case such a formula has been written by Sklyanin [10]. A trigonometric difference version appears in Tarasov and Varchenko [12].

3.3. **Restrictions.** The linear difference operators defined in the previous subsections act on meromorphic funtions of complex variables $\lambda, x_1, \ldots, x_n$. To compare these operators with evaluation modules of the elliptic quantum groups and with transfer matrices of IRF models, we have to restrict their action to the space of meromorphic function defined on submanifolds of $\mathbb{C}^{n+1}$. The conditions for a difference operator X with meromorphic coefficients to be defined on meromorphic functions on a submanifold S are that the value at a generic $x \in S$ of $Xf(x)$ is well-defined (i.e., there are no poles at generic points of S) and is only a function of the values of f at points of S. Equivalenty, a difference operator X can be restricted to S if it maps functions vanishing on S to functions vanishing on S. The restriction is then identified with the induced action on the quotient by the function vanishing on S.

These conditions are fulfilled in the following situations:

1. *Restriction to discrete values of* x_i. We assume that z_i, η are generic and that the Λ_i are non-negative integers. We take S to be the set

$$\begin{aligned} S_0 \;=\; & \{(\lambda, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \mid -x_i = z_i + \eta(\Lambda_i - 2m_i), \\ & m_i = 0, 1, \ldots, \Lambda_i, i = 1, \ldots, n\}. \end{aligned}$$

Since the steps in the difference operators are by multiples of 2η, it is clear that one can restrict the action of $a(z), \ldots, d(z)$ to functions on subsets given by conditions $x_i \in a_i + 2\eta\mathbb{Z}$, for generic a_i, η. The genericity condition on a_i, η is that the poles at $x_i - x_j \bmod \Gamma$ of the coefficients of the difference operators are never on the subset. What we have to check is that the restriction to these finite sets of values for x_i is well-defined. Since only $b(z)$, $c(z)$ shift the value of x_i, it is sufficient to consider these two operators. For a function $f(\lambda, x_1, \ldots, x_n)$, the value at $-x_i = z_i + \eta\Lambda_i$ of $b(z)f$ appears to depend on the value of f at $-x_i + 2\eta = z_i + \eta\Lambda_i + 2\eta$, which is not in S_0, but in fact it does not, since the coefficient $\Delta_-(-x_i)$ vanishes there. Similarly $c(z)$ is well-defined on S_0.

For this restriction h has discrete spectrum. It will be used to compare our representation with tensor products of irreducible representations.

2. *Restriction to $\lambda = \eta h$.* Let S be the set
$$S_1 = \{(\lambda, x_1, \ldots, x_n) \,|\, \lambda = -\sum_i (x_i + z_i)\}.$$
Then $b(z), c(z)$ can be restricted to functions on S_1. Indeed, if u is a function vanishing on S_1, then $T_{x_i}^{-2\eta} T_\lambda^{2\eta} u$ still vanishes on S_1. The denominators of the coefficients of $b(z)$ do not vanish at generic points of S_1. Thus $b(z)u = 0$ if u vanishes on S_1. The same reasoning applies to $c(z)$.

This restriction is needed, as we shall see, to construct commuting transfer matrices.

3. *IRF restriction.* Consider the restriction of $b(z), c(z)$ on functions on $S = S_0 \cap S_1$. If z_i and η are generic, the only possible pole on S in the coefficients of these differential operators comes from the denominator $\theta(\lambda)$. This denominator does not vanish if we assume for instance that the Λ_i are all odd.

Here S is finite, so that the restriction is to a finite dimensional space of functions, which will be identified with the space of states of an IRF model.

3.4. **Commuting difference operators.** One of the main properties of R-matrices to statistical mechanics is that L-operators obeying quantum group relations give rise to commuting transfer matrices $\mathrm{tr}_V L$. In [10] it is noticed that more generally one can consider $\mathrm{tr}_V((K \otimes 1)L)$ for some endomorphism of V such that $K \otimes K$ commutes with the R-matrices.

In our dynamical case it is known that the traces $a(z) + d(z)$ commute for different values of z when acting on the zero weight space of a module over $E_{\tau,\eta}(sl_2)$. Another possibility to obtain commuting operators is to take the *twisted transfer matrix* $\mathrm{tr}_V((K \otimes 1)L)$ with a suitable K:

Proposition 3.2. *For any $\vartheta \in \mathbb{C}$, the operators $T(z) = b(z) + \vartheta c(z)$, $z \in \mathbb{C}$, restricted to functions on the submanifold S_1 given by the equation $\lambda + \sum_{i=1}^n x_i + \sum_{i=1}^n z_i = 0$, form a commuting family.*

This proposition can be proved directly or by the following general argument.

It is first of all sufficient to consider the case $\vartheta = 1$ since the other cases are obtained by conjugating the operator by the multiplication by an exponential function of λ. Then we may write $T(z)$ as

$$T(z) = \sum_{\mu=\pm 1} \mathrm{tr}_{V[\mu]}(K \otimes 1\, L(z, \lambda)) T_\lambda^{-2\eta\mu},$$

with

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The partial trace $\mathrm{tr}_{V[\mu]} : \mathrm{End}(V \otimes W) \to \mathrm{End}(W)$ is the homomorphism such that if $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$, then $\mathrm{tr}_{V[\mu]}(a \otimes b) = \sum e^i(ae_i)b$ for any basis (e_i) of $V[\mu]$ with $e^i \in V^*$ defined by $e^i(e_j) = \delta_{ij}$ and $e^i(w) = 0$ for $w \in V[\nu]$, $\nu \neq \mu$. We have

$$K \otimes K\, R(z, \lambda) = R(z, -\lambda)\, K \otimes K. \tag{9}$$

We then write the RLL relations in the form

$$\begin{gathered} R^{(12)}(z - w, \lambda - 2\eta h) L^{(1)}(z, \lambda) L^{(2)}(w, \lambda - 2\eta h^{(1)}) R^{(12)}(z - w, \lambda)^{-1} = \\ L^{(2)}(w, \lambda) L^{(1)}(z, \lambda - 2\eta h^{(2)}), \end{gathered}$$

multiply both sides by $(K \otimes K)^{(12)}$ from the left, and take the partial trace over a weight space $(V \otimes V)[\mu]$. Using (9), we obtain

$$\begin{gathered} \mathrm{tr}_{(V \otimes V)[\mu]}(R^{(12)}(z - w, -\lambda + 2\eta h) K^{(1)} K^{(2)} \\ L^{(1)}(z, \lambda) L^{(2)}(w, \lambda - 2\eta h^{(1)}) R^{(12)}(z - w, \lambda)^{-1}) = \\ \mathrm{tr}_{(V \otimes V)[\mu]} K^{(2)} L^{(2)}(w, \lambda) K^{(1)} L^{(1)}(z, \lambda - 2\eta h^{(2)}). \end{gathered}$$

The next step is to use the cyclicity of the trace to bring the first R matrix to the right. For this we need the commutation relations of h with the product in the trace. Since $[L^{(i)}, h^{(i)} + h] = 0$ and $h^{(i)} K^{(i)} = -K^{(i)} h^{(i)}$, we see that

$$h\, \mathrm{tr}_{(V \otimes V)[\mu]}(A^{(12)} K^{(1)} K^{(2)} L^{(1)} L^{(2)}) = \mathrm{tr}_{(V \otimes V)[\mu]}(A^{(12)} K^{(1)} K^{(2)} L^{(1)} L^{(2)}) \cdot (h + 2\mu),$$

for any $A \in \mathrm{End}(V \otimes V)$ commuting with $h^{(1)} + h^{(2)}$. We then get

$$\begin{gathered} \textstyle\sum_\mu \mathrm{tr}_{(V \otimes V)[\mu]}(K^{(1)} K^{(2)} L^{(1)}(z, \lambda) L^{(2)}(w, \lambda - 2\eta h^{(1)}) \\ R^{(12)}(z - w, \lambda)^{-1} R^{(12)}(z - w, -\lambda + 2\eta(h + 2\mu)) T_\lambda^{-2\eta\mu} = \\ T(w) T(z) \end{gathered}$$

If we then have a relation $\eta h\, u(\lambda) = \lambda u(\lambda)$, and we apply the above equation to u, we may replace h in the left-hand side by $\lambda/\eta - 2\mu$, and the R matrices cancel, so that $T(z)T(w) = T(w)T(z)$.

3.5. **Evaluation modules.** Here we show that the restriction of the operators a, b, c, d of Prop. 3.1 to functions on the submanifold S_0 is essentially the tensor product of finite dimensional irreducible evaluation modules of [7].

Proposition 3.3. *Suppose that $\eta, z_1, \dots, z_n$ are generic. Let a, b, c, d be the difference operators defined in 3.2 restricted to functions on S_0 and let $\kappa(z) = \prod_{i=1}^n \theta(z-z_i-\eta\Lambda_i)^{-1}$. Then $\bar{a}(z) = \kappa(z)a(z), \dots, \bar{d}(z) = \kappa(z)d(z)$ define an $E_{\tau,\eta}(sl_2)$ module isomorphic to the tensor product $L_{\Lambda_1}(z_1-\eta) \otimes \cdots \otimes L_{\Lambda_n}(z_n-\eta)$ of irreducible evaluation modules.*

Let W be the space of functions on S. It is a vector space over the field of meromorphic functions of λ of dimension $\prod_{i=1}^n(\Lambda_i+1)$. To prove this proposition we have to identify a highest weight vector in v, i.e., an eigenvector of $a(z)$, $d(z)$ and h killed by $c(z)$. The eigenvalues $(A(z,\lambda), D(z,\lambda), \Lambda)$ of $(a(z), d(z), h)$ determine then by [7] uniquely an irreducible module up to isomorphism.

Let $\delta_a(x_i) \in W$ be the delta function at $x_i = a$: $\delta_a(x_i) = 1$ if $x_i = a$, $\delta_a(x_i) = 0$, if $x_i \neq a$. The highest weight vector may be taken as the product of delta functions

$$v_{h.w.} = \prod_{i=1}^n \delta_{-z_i-\Lambda_i\eta}(x_i)\,.$$

This function is indeed annhilated by $c(z)$ and $hv_{h.w.} = \sum \Lambda_i v_{h.w.}$. Moreover $v_{h.w.}$ is an eigenvector for $a(z)$ and $d(z)$:

$$\begin{aligned} a(z)v_{h.w.} &= \prod_{i=1}^n \theta(z-z_i-\Lambda_i\eta)\frac{\theta(\lambda - \sum_{i=1}^n \Lambda_i\eta + \sum_{i=1}^n \Lambda_i\eta)}{\theta(\lambda)} v_{h.w.} \\ &= \prod_{i=1}^n \theta(z-z_i-\Lambda_i\eta)v_{h.w.}\,. \end{aligned}$$

Thus $a(z)v_{h.w.} = A(z,\lambda)v_{h.w.}$, with $A(z,\lambda) = \prod_{i=1}^n \theta(z-z_i-\Lambda_i\eta)$. Similary $d(z)v_{h.w.} = D(z,\lambda)v_{h.w.}$ with eigenvalue

$$\begin{aligned} D(z,\lambda) &= \frac{\theta(\lambda - 2\eta\sum_{i=1}^n \Lambda_i)}{\theta(\lambda)} Det(z-2\eta,\lambda)A^{-1}(z-2\eta,\lambda+2\eta)\,. \\ &= \frac{\theta(\lambda - 2\eta\sum_{i=1}^n \Lambda_i)}{\theta(\lambda)} \prod_{i=1}^n \frac{\theta(z-z_i-\Lambda_i\eta-2\eta)\theta(z-z_i+\Lambda_i\eta)}{\theta(z-z_i-\Lambda_i\eta-2\eta)}\,. \end{aligned}$$

Thus the eigenvalues of $(\bar{a}(z), \bar{d}(z), h)$ are $(1, \bar{D}(z,\lambda), \sum\Lambda_i)$ with

$$\bar{D}(z,\lambda) = \frac{\theta(\lambda - 2\eta\sum_{i=1}^n \Lambda_i)}{\theta(\lambda)} \prod_{i=1}^n \frac{\theta(z-z_i+\Lambda_i\eta)}{\theta(z-z_i-\Lambda_i\eta)}\,,$$

which indeed reproduces the highest weight defined in [7], p. 750.

3.6. **Separation of variables: continuous case.** In this section we find an analogue of the results of 2.4, 2.5. We consider the continuous case, in which the variables x_i take arbitrary complex values. Then the eigenvalue problem for a function u on S_1 reads $T(z)u(x) = \epsilon(z)u(x)$, with

$$\begin{aligned} T(-z)u(x_1,\dots,x_n) \;\; = \;\; & \sum_{i=1}^{n} \frac{\theta(\lambda - z + x_i)}{\theta(\lambda)} \prod_{j\neq i} \frac{\theta(z-x_j)}{\theta(x_i-x_j)} \\ & \Big(\prod_{k=1}^{n} \theta(x_i + z_k + \eta\Lambda_k) u(x_1,\dots,x_i - 2\eta,\cdots,x_n) \\ & + \prod_{k=1}^{n} \theta(x_i + z_k - \eta\Lambda_k) u(x_1,\dots,x_i + 2\eta,\dots,x_n)\Big)\,, \end{aligned}$$

where we view a function u on S_1 as a function of $x_1,\dots,x_n$ by setting $\lambda = -\sum(x_j + z_j)$.

From this formula it is clear that $T(z)u(x)$ is an entire holomorphic function of z with theta function behavior as z is translated by elements of the lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. It follows that a necessary condition of $\epsilon(z)$ to be an eigenvalue is that ϵ belong to the space $Theta_n(\chi_0)$ of theta functions of order n with character $\chi_0 : \Gamma \to \mathbb{C}^*$ (see the Appendix) given by $\chi_0(r + s\tau) = (-1)^{n(r+s)} \exp(2\pi i \sum z_k)$. This means that ϵ is an entire function obeying

$$\epsilon(z+1) = \chi_0(1)\epsilon(z), \qquad \epsilon(z+\tau) = \chi_0(\tau)e^{-\pi i n(2z+\tau)}\epsilon(z).$$

The method of separation of variables consists in looking for common eigenfunctions $u(x)$ in the factorized form $u(x) = \prod_{i=1}^n Q(x_i)$. Setting $z = -x_i$ in the eigenvalue problem $(T(z) - \epsilon(z))\,u = 0$ we see that a necessary condition is that Q, ϵ obey the difference equation

$$A_+(x)\,Q(x-2\eta) + A_-(x)\,Q(x+2\eta) = \epsilon(-x)Q(x), \qquad A_\pm(x) = \prod_{k=1}^{n} \theta(x + z_k \pm \eta\Lambda_k)$$

As explained in the Appendix, this difference equation has an *elliptic polynomial solutions*, i.e., a solution of the form

$$Q(x) = e^{ax} \prod_{k=1}^{m} \theta(x - w_k). \tag{10}$$

if $\sum \Lambda_i$ is an even integer $2m$. Such a solution may be constructed by the Bethe ansatz:

Proposition 3.4. *Suppose that $\Lambda_1 + \cdots + \Lambda_n = 2m$ for some positive integer m, and let $(a, w_1, \dots, w_m)$ be a solution of the system of Bethe ansatz equations*

$$A_+(w_i) \prod_{j:j\neq i} \theta(w_i - w_j - 2\eta) = e^{4a\eta} A_-(w_i) \prod_{j:j\neq i} \theta(w_i - w_j + 2\eta) \qquad i = 1,\dots,m, \tag{11}$$

such that $w_i \neq w_j \mod \Gamma, (i \neq j)$. Then $u = \prod Q(x_i)$ with $Q(x) = e^{ax} \prod_{k=1}^{m} \theta(x - w_k)$ is a common eigenfunction of $T(z)$.

Proof: This is a rephrasing of the first part of Prop. A.4. □

Definition. An eigenfunction of the form of Prop. 3.4 is called Bethe eigenfunction.

Conversely, let us suppose that $\sum \Lambda_i = 2m$, $m \in \mathbb{Z}_{>0}$ and show that all eigenfunctions in a suitable class are of this form. Let, for a character $\chi : \Gamma \to \mathbb{C}^*$, $\mathcal{H}_m(\chi)$ be the space of meromorphic functions of n complex variables $x_1, \dots, x_n$ such that

$$\begin{aligned} u(\cdots, x_i + 1, \cdots) &= \chi(1) u(\cdots, x_i, \cdots), \\ u(\cdots, x_i + \tau, \cdots) &= \chi(\tau) e^{-\pi i m(2x_j + \tau)} u(\cdots, x_i, \cdots), \end{aligned}$$

The following result can then easily be verified using the behavior of the coefficients of the difference operator $T(z)$.

Lemma 3.5. *For any character χ and any $z \in \mathbb{C}$, $T(z)$ preserves $\mathcal{H}_m(\chi)$.*

Let for a character $\chi \in \Gamma^*$, $\Sigma(\chi) \subset \Theta_m(\chi_0)$ be the set of functions ϵ so that there exists a *holomorphic* common eigenfunction $u \in \mathcal{H}_m(\chi)$ of $T(z)$, $z \in \mathbb{C}$, with eigenvalue $\epsilon(z)$.

Theorem 3.6. *Suppose that $\Lambda_1 + \cdots + \Lambda_n = 2m$ for some positive integer m. Then $\epsilon \in \Sigma(\chi)$ if and only if there is an eigenfunction $u(x)$ of the form $u(x) = \prod_{i=1}^n Q(x_i)$ with eigenvalue ϵ, such that $Q(x) = e^{ax} \prod_{i=1}^m \theta(x - w_i)$ for some solution $(a, w_1, \dots, w_m)$ of the Bethe ansatz equations with*

$$\chi(1) = (-1)^m e^a, \qquad \chi(\tau) = (-1)^m e^{a\tau + 2\pi i \sum w_k}.$$

Proof: It remains to show that if $\epsilon \in \Sigma(\chi)$ then this eigenvalue corresponds to a Bethe eigenfunction.

Suppose that $\epsilon \in \Sigma(\chi)$ and u is a holomorphic eigenfunction in $\mathcal{H}_m(\chi)$ with this eigenvalue. Then, for each i, the function u viewed as a function of x_i belongs to $\Theta_m(\chi)$. By setting $z = -x_i$ in the eigenvalue equation $T(z)u(x) = u(x)$, we see that u is a solution of the separated equation

$$\prod_{k=1}^n \theta(x_i + z_k + \eta)\, T_{x_i}^{-2\eta} u(x) + \theta(x_i + z_k - \eta)\, T_{x_i}^{2\eta} u(x) = \epsilon(-x_i) u(x). \tag{12}$$

Thus this equation admits a non-trivial solution $Q(x_i)$ in $\Theta_m(\chi)$ (we consider the remaining variables x_j, $j \neq i$ as fixed). By the factorization theorem for theta function (see the Appendix), such a solution is, up to normalization, of the form $Q(x_i) = e^{ax} \prod_{j=1}^m \theta(x_i - w_j)$, and χ is related to a and w_k by the equations stated in the Theorem. Setting $x_i = w_j$ in (12) gives then the Bethe ansatz equations. □

4. IRF models with antiperiodic boundary conditions

We consider here the case where $\Lambda_1 = \cdots = \Lambda_n = 1$ and show that the commuting transfer matrices $T(z)$ restricted to functions on $S_0 \cap S_1$ are transfer matrices of IRF models with antiperiodic boundary conditions. The IRF (interaction-round-a-face) models (see [2], [1], [3]) of statistical mechanics are two-dimensional lattice models in which the configurations over which one sums in the partition function associate an element, the height, of a certain set to each pair of neighboring points. The weight of a configuration is the product of local Boltzmann weights associated to each "face", or square formed by four neighboring points. The local Boltzmann weight $W(a, b, c, d|z)$ depends on the heights a, b, c, d at the neighboring points and the spectral parameter $z \in \mathbb{C}$. The spectral parameter can be any fixed number, but more generally, in the inhomogeneous model, one associates a spectral parameter to each row and column of faces in the lattice and takes z to be the difference between the row parameter and the column parameter. In the simplest sl_2 case the heights are real numbers and the local Boltzmann weights are related to matrix elements of our dynamical R-matrix by

$$R(z, \lambda = -2\eta d)\, e[c-d] \otimes e[b-c] = \sum_a W(c, b, a, d|z)\, e[b-a] \otimes e[a-d], \tag{13}$$

where $c-d, b-c, b-a, a-d \in \{-1, 1\}$. If the latter conditions are not fulfilled, then $W = 0$.[2]

As shown in [8], the transfer matrix $a(z) + d(z)$ on the zero weight subspace of the tensor product of two dimensional representations is then identified with the row-to-row transfer matrix of the IRF model with periodic boundary conditions. Here we show that a similar computation applied to the twisted transfer matrix $b(z) + c(z)$ gives the row-to-row transfer matrix of the IRF model with antiperiodic boundary conditions.

The formulae are as follows: Let $V(z_1) \otimes \cdot \otimes V(z_n)$ be a tensor product of two dimensional evaluation modules and assume that n is odd. Then the L-operator for this module is

$$L(z, \lambda) = R^{(01)}(z - z_1, \lambda - 2\eta \sum_{i=2}^{n} h^{(i)}) \cdots R^{(0n-1)}(z - z_{n-1}, \lambda - 2\eta h^{(n)}) R^{(0n)}(z - z_n, \lambda).$$

Then the twisted transfer matrices, see 3.4, acting on functions $u(\lambda)$ restricted to the submanifold given by the equation $\lambda = \eta h$ form a commuting family parametrized by the spectral parameter. Let us call these transfer matrices $T_{\mathrm{IRF}}(z)$. The restriction means that $T_{\mathrm{IRF}}(z)$ preserves functions $u(\lambda)$ of the form $u(\lambda) = \sum_\mu \delta_{\eta\mu}(\lambda) u[\mu]$ for some vectors $u[\mu]$ of weight μ. Here $\delta_a(b)$ is one if $a = b$ and zero otherwise. The restriction to this subspace of function of λ is well-defined if n is odd, as the singularities of the L operator are at $\lambda \in 2\eta\mathbb{Z}$ whereas λ takes values $\eta\mu$, where μ, the weight of a vector in $\otimes V(z_i)$ is an odd number if n is odd. So if η is generic, there are no singularities at this points.

[2] The normalization chosen here is so that $W(l, l+1, l+2, l+1|z) = 1$ rather than the more common $\theta(z - 2\eta)/\theta(2\eta)$. Also our spectral parameter is normalized in a different way than in the literature.

Explicitly,

$$T_{\mathrm{IRF}}(z) = \bar{b}(z) + \bar{c}(z) =$$

$$\sum_\mu \mathrm{tr}^{(0)}_{V[\mu]} K^{(0)} R^{(01)}(z-z_1, \lambda - 2\eta \textstyle\sum_{i=2}^n h^{(i)}) R^{(02)}(z-z_2, \lambda - 2\eta \sum_{i=3}^n h^{(i)}) \cdots$$

$$R^{(0n-1)}(z-z_{n-1}, \lambda - 2\eta h^{(n)}) R^{(0n)}(z-z_n, \lambda) T_\lambda^{-2\eta\mu}, \tag{14}$$

with

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

The product of R-matrices acts on $V \otimes (\otimes_{i=1}^n V)$, where the index (0) refers to action on the first factor of the tensor product, $\mathrm{tr}^{(0)}$ denotes the trace over the first factor. The subsequent factors are numbered accordingly from 1 to n.

The twisted transfer matrices are defined on a 2^n-dimesnional space. This space has a basis $\delta_{\eta \sum \sigma_i}(\lambda) e[\sigma_1] \otimes \cdots \otimes e[\sigma_n]$ with $\sigma_i \in \{1, -1\}$. It is convenient to write this basis in terms of antiperiodic paths: Let for $a_1, \ldots, a_n, a_{n+1} \in \mathbb{Z} + n/2$, such that $|a_i - a_{i+1}| = 1$ $(i = 1, \ldots, n)$, and $a_{n+1} = -a_1$

$$|a_1, \ldots, a_{n+1}\rangle = \delta_{-2\eta a_{n+1}}(\lambda) e[a_1 - a_2] \otimes \cdots \otimes e[a_n - a_{n+1}]$$

Proposition 4.1. *For any antiperiodic path $|a_1, \ldots, a_n, a_{n+1} = -a_1\rangle$, we have*

$$T_{\mathrm{IRF}}(z) \quad |a_1, \ldots, a_{n+1}\rangle \quad = \quad \sum_{b_1, \ldots, b_n, b_{n+1} = -b_1} \prod_{i=1}^n W(a_{i+1}, a_i, b_i, b_{i+1}) \quad |b_1, \ldots, b_{n+1}\rangle\,, \tag{15}$$

This expression is, by definition, the transfer matrix of IRF models with antiperiodic boundary conditions. The partition function for the IRF model with antiperiodic boundary conditions in one direction is then $\mathrm{tr} T(w_1) \cdots T(w_m)$. the w_i are the spectral parameters associated to the m rows, and the z_i are the spectral parameters associated to the n columns.

Proof of the proposition: Let $e^*[1]$, $e^*[-1]$ be the dual basis to $e[1], e[-1]$.

$$T_{\mathrm{IRF}}(z) \quad |a_1, \ldots, a_{n+1}\rangle \quad =$$

$$\sum_\mu \mathrm{tr}^{(0)}_{V[\mu]} K^{(0)} R^{(01)}(z-z_1, \lambda - 2\eta \sum_{i=2}^n h^{(i)}) R^{(02)}(z-z_2, \lambda - 2\eta \sum_{i=3}^n h^{(i)})$$

$$\cdots R^{(0n)}(z-z_n, \lambda)\, \delta_{-2\eta(a_{n+1}-\mu)}(\lambda) e[a_1 - a_2] \otimes \cdots \otimes e[a_n - a_{n+1}] =$$

$$\sum_{\mu} e^{(0)\,*}[\mu]K^{(0)}R^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n}h^{(i)})\cdots R^{(0n)}(z-z_n,\lambda)\delta_{-2\eta(a_{n+1}-\mu)}(\lambda)$$
$$e^{(0)}[\mu]\otimes e[a_1-a_2]\otimes\cdots\otimes e[a_n-a_{n+1}]=$$
$$\sum_{b_{n+1}} e^{(0)\,*}[a_{n+1}-b_{n+1}]K^{(0)}R^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n}h^{(i)})\cdots R^{(0n)}(z-z_n,\lambda)$$
$$\delta_{-2\eta b_{n+1}}(\lambda)e^{(0)}[a_{n+1}-b_{n+1}]\otimes e[a_1-a_2]\otimes\cdots\otimes e[a_n-a_{n+1}]=$$
$$\sum_{b_{n+1},b_n} e^{(0)\,*}[a_{n+1}-b_{n+1}]K^{(0)}R^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n}h^{(i)})\cdots$$
$$R^{(0n-1)}(z-z_{n-1},\lambda-2\eta h^{(n)})\,W(a_{n+1},a_n,b_n,b_{n+1})\delta_{-2\eta b_{n+1}}(\lambda)$$
$$e^{(0)}[a_n-b_n]\otimes e[a_1-a_2]\otimes\cdots\otimes e[a_{n-1}-a_n]\otimes e[b_n-b_{n+1}]=\cdots=$$
$$\sum_{b_{n+1},\ldots b_i} e^{(0)\,*}[a_{n+1}-b_{n+1}]K^{(0)}R^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n}h^{(i)})\cdots$$
$$R^{(0i-1)}(z-z_{i-1},\lambda-2\eta\sum_{j=i}^{n}h^{(j)})\,\Pi_{j=0}^{n-i}W(a_{n+1-j},a_{n-j},b_{n-j},b_{n+1-j})$$
$$\delta_{-2\eta b_{n+1}}(\lambda)e^{(0)}[a_i-b_i]\otimes$$
$$e[a_1-a_2]\otimes e[a_{i-1}-a_i]\otimes e[b_i-b_{i+1}]\otimes\cdots\otimes e[b_n-b_{n+1}]=\cdots=$$
$$\sum_{b_{n+1},\ldots,b_1} e^{(0)\,*}[a_{n+1}-b_{n+1}]K^{(0)}\Pi_{i=1}^{n}W(a_{i+1},a_i,b_i,b_{i+1})$$
$$\delta_{-2\eta b_{n+1}}(\lambda)e^{(0)}[a_1-b_1]\otimes e[b_1-b_2]\otimes\cdots\otimes e[b_n-b_{n+1}]=$$
$$\sum_{b_{n+1},\ldots,b_1} e^{(0)\,*}[a_{n+1}-b_{n+1}]e^{(0)}[-a_1+b_1]\otimes e[b_1-b_2]\otimes\cdots\otimes e[b_n-b_{n+1}]$$
$$\delta_{-2\eta b_{n+1}}(\lambda)\,\Pi_{i=1}^{n}W(a_{i+1},a_i,b_i,b_{i+1})\,.$$

By evaluating the linear form $e^*[a_{n+1}-b_{n+1}]$ we see that only the terms with $b_{n+1}=-b_1$ contribute to the sum. The proof is complete.

4.1. **Separation of variables for IRF models.** Let us consider the eigenvalue problem for the transfer matrix restricted to $S_0\cap S_1$ for $\Lambda_1=\cdots=\Lambda_n=1$ and n odd. The eigenfunction may be viewed as a function $u(x_1,\ldots,x_n)$ defined for $x_i\in$

$\{-z_i-\eta,-z_i+\eta\}$, $1\le i\le n$. The eigenvalue problem reads $T(z)u(x)=\epsilon(z)u(x)$, with

$$
\begin{aligned}
T(-z)u(x_1,\dots,x_n) &= \sum_{i=1}^n \frac{\theta(\lambda-z+x_i)}{\theta(\lambda)}\prod_{j\neq i}\frac{\theta(z-x_j)}{\theta(x_i-x_j)} \\
&\quad \Big(\prod_{k=1}^n \theta(x_i+z_k+\eta\Lambda_k)u(x_1,\cdots,x_i-2\eta,\cdots,x_n) \\
&\quad +\prod_{k=1}^n \theta(x_i+z_k-\eta\Lambda_k)u(x_1,\dots,x_i+2\eta,\dots,x_n)\Big)\,,
\end{aligned} \tag{16}
$$

where $\lambda=-\sum_{k=1}^n(x_k+z_k)$.

Theorem 4.2. *Suppose that $\Lambda_1=\cdots=\Lambda_n=1$ with n odd, $\eta\notin\Gamma$, and that $z_i\neq z_j+2\eta\ell$ mod Γ for $i\neq j$ and $\ell=0,\pm1$. Let $T(z)=b(z)+c(z)$ be the transfer matrix restricted to the 2^n dimensional space of functions on $S_0\cap S_1$. Then a function $\epsilon(z)$ is a common eigenvalue of the transfer matrices $T(z)$, $z\in\mathbb{C}$, if and only if*

(i) *$\epsilon\in\Theta_n(\chi)$ with $\chi(1)=(-1)^n$, $\chi(\tau)=(-1)^n e^{2\pi i\sum z_j}$ and*

(ii) *ϵ obeys the quadratic relations*

$$
\epsilon(z_i-\eta)\epsilon(z_i+\eta)=\prod_{k=1}^n\theta(z_k-z_i+2\eta)\theta(z_k-z_i-2\eta),\qquad i=1,\dots,n. \tag{17}
$$

To prove this theorem, let $S=\times_{i=1}^n\{-z_i-\eta,-z_i+\eta\}$. Let us first assume that $u(x)$, $x\in S$ is a common eigenfunction of $T(z)$ with eigenvalue $\epsilon(z)$. In particular u does not vanish identically on S. From the transformation properties of $T(z)$ under shifts of z by Γ we see that $\epsilon(z)$ has to belong to $\Theta_n(\chi)$. Then setting $z=-x_i$ in the eigenvector equation $T(z)u(x)=\epsilon(z)u(x)$, we get the separated equation

$$
\prod_{k=1}^n\theta(x_i+z_k+\eta)\,T_{x_i}^{-2\eta}u(x)+\theta(x_i+z_k-\eta)\,T_{x_i}^{2\eta}u(x)=\epsilon(-x_i)u(x). \tag{18}
$$

Inserting the two values of x_i yields

(19)

$$
\prod_{k=1}^n\theta(z_k-z_i+2\eta)\,u(x_1,\dots,-z_i-\eta,\dots,x_n)=\epsilon(z_i-\eta)u(x_1,\dots,-z_i+\eta,\dots,x_n),
$$

$$
\prod_{k=1}^n\theta(z_k-z_i-2\eta)\,u(x_1,\dots,-z_i+\eta,\dots,x_n)=\epsilon(z_i+\eta)u(x_1,\dots,-z_i-\eta,\dots,x_n).
$$

By multiplying these two equations we see that ϵ must obey the identity (17), provided we can prove that, for at least one choice of x_j, the product

$$
u(x_1,\dots,-z_i-\eta,\dots,x_n)\,u(x_1,\dots,-z_i+\eta,\dots,x_n)
$$

does not vanish. This follows from the fact that u is not identically zero, so that at least one factor of this product is nonzero, and that the product of theta functions on the

left-hand side of (19) is not zero with our assumption on the z_j, so that also the other factor is nonzero.

We have thus shown that a necessary condition for a funtion $\epsilon(z)$ to be a common eigenvalue is that ϵ is a theta funtion obeying the quadratic relations (17).

Conversely, let us suppose that ϵ obeys (17). Then, for every i, the system of equations

$$\prod_{k=1}^{n} \theta(x + z_k + \eta)\, Q(x - 2\eta) + \theta(x + z_k - \eta)\, Q(x + 2\eta) = \epsilon(-x)Q(x), \quad x = -z_i \pm \eta,$$

admits a non-trivial solution $Q_i(x)$. It follows that $u(x) = \prod_{i=1}^{n} Q_i(x_i)$ obeys (18). Thus, for any $x \in S$, $(T(z) - \epsilon(z))u(x)$, viewed as a function of z is a theta function in $\Theta_n(\chi)$ vanishing at n distinct points $-x_1, \ldots, -x_n$. As explained in the Appendix, this implies that that either the function vanishes identically or $-\sum x_i = \sum z_i$. Since for $x \in S$, $\sum x_i$ is $-\sum z_i$ plus an *odd* multiple of η, the latter alternative cannot hold. Thus u is an eigenfuntion with eigenvalue ϵ. The Theorem is proven.

5. Conclusions

In this paper we have studied integrable models associated to elliptic curves by Sklyanin's method of separation of variables. We have shown how in the elliptic version of the Gaudin model, previously considered in [4], this method implies the completeness of Bethe eigenfunction in Verma modules, in the sense that for every eigenvalue one has a Bethe eigenfunction (although in the case of degenerate eigenvalues one cannot exclude the existence of additional eigenfunction not of Bethe type).

In the difference case, the analogue of the transfer matrix in the separated variables was shown to be a twisted transfer matrix associated to a representation of $E_{\tau,\eta}(sl_2)$ by difference operators. The eigenvalue problem can be posed in two different cases. In the continuous case, the variables are assumed to be complex and one asks for eigenfunctions with theta functions properties, as one does in the differential case. These eigenfunctions are of Bethe ansatz type. In the discrete case, the variables are assumed to take values in a finite set. The eigenvalue problem can still be solved by separating variables yielding eigenfunctions of the (inhomogeneous) row-to-row transfer matrix of IRF models with antiperiodic boundary conditions. In both cases one has a completeness result.

Let us conclude by mentioning some open questions. In the difference case one has the transfer matrix in the "separated variables". Is there a "quantum Radon transform" as in the differential case, which maps the transfer matrix of the IRF model with periodic boundary conditions to this transfer matrix? Is there an analogue of the equivalence between the local problem and the global problem of [12] in the elliptic case?

Appendix A. Elliptic polynomials

A.1. **Theta functions.** Let $\operatorname{Im}\tau > 0$ and set $q = e^{2\pi i\tau}$. Let Γ be the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and $\Gamma^* \simeq (\mathbb{C}^\times)^2$ the group of group homomorphisms $\Gamma \to \mathbb{C}^\times$. Let ϕ be the homomorphism

$\phi : \chi \mapsto \frac{1}{2\pi i}(\ln\, \chi(\tau) - \tau \ln\, \chi(1))$ (or, more invariantly, $\frac{1}{2\pi i}(\omega_1 \ln\, \chi(\omega_2) - \omega_2 \ln\, \chi(\omega_1))$, for any oriented basis ω_1, ω_2 of Γ) from Γ^* to the elliptic curve $E = \mathbb{C}/\Gamma$.

For $\chi \in \Gamma^*$ let $\Theta_k(\chi)$ be the space of theta functions of level k and character χ. It consists of entire holomorphic functions $f(z)$ such that $f(z + r + s\tau) = \chi(r + s\tau)\exp(-\pi i k(s^2\tau + 2sz))f(z)$ for all $r + s\tau \in \Gamma$.

The dimension of $\Theta_k(\chi)$ is zero if $k < 0$. It is k if $k \geq 1$. The dimension of $\Theta_0(\chi)$ is one if $\phi(\chi) = 0$ and vanishes otherwise.

We have the unique factorization result:

Proposition A.1. *The function of $z \in \mathbb{C}$*

$$f(a, w; z) = e^{az} \prod_{j=1}^{k} \theta(z - w_j)$$

belongs to $\Theta_k(\chi)$, with $\chi(r + s\tau) = (-1)^{(r+s)k} e^{ra+s(a\tau+2\pi i \sum_j w_j)}$. Every function in $\Theta_k(\chi)$ is of the form $C \cdot f(a, w; z)$ for some constant C and this representation is unique up to permutation of the w_j if one requires the w_j to be in the fundamental domain $F = \{x + y\tau \,|\, x, y \in [0, 1)\}$.

Proof: It follows from the transformation properties of theta functions that the number of zeros $(2\pi i)^{-1} \int_{\partial F} d \ln g$, counted with multiplicities, in F of a theta function $g \in \Theta_k(\chi)$ is k. If $w_1, \dots, w_k$ denote the zeros of g then $g(z)/f(a, w; z)$ is doubly periodic and regular, thus constant. Uniqueness follows from the fact that a is uniquely determined by the zeros w_j and the character χ. $\square$

Corollary A.2. *Let E be the elliptic curve $\mathbb{C}/\Gamma$, and, for $k = 1, 2, \dots$, $S^k(E) = E/S_k$ be its k^{th} symmetric power. The map $\mathbb{P}(\Theta_k(\chi)) \to S^k(E)$ sending a function to the set of its zeros modulo Γ, is injective. Its image consists of classes $[w_1, \dots, w_k]$ such that $\sum_j w_j = \phi(\chi) + k\delta$, where δ is the image in E of $(1 + \tau)/2$.*

A.2. **Interpolation.**

Theorem A.3. *Suppose $z_1, \dots, z_k \in \mathbb{C}$ are pairwise distinct modulo Γ and $\chi \in \Gamma^*$ is such that $\sum_{i=1}^{k} z_i \neq \phi(\chi) + k\delta \mod \Gamma$. Then for any $y_1, \dots, y_k \in \mathbb{C}$ there exists a unique function $f \in \Theta_k(\chi)$ such that $f(z_i) = y_i$ for all $i = 1, \dots, k$.*

The interpolation formula giving f is

$$f(z) = \sum_{j=1}^{m} y_j e^{2\pi i a(z - z_j)} \frac{\theta(z - z_j + b)}{\theta(b)} \prod_{l : l \neq j} \frac{\theta(z - z_l)}{\theta(z_j - z_l)}.$$

Here a and b are such that the character of all terms in the sum are equal to χ, so

$$\begin{aligned} a &= \frac{1}{2\pi i} \ln \chi(1) - \frac{k}{2}, \\ b &= \frac{1}{2\pi i} \left(\tau \ln \chi(1) - \ln \chi(\tau)\right) + \sum_{j=1}^{k} z_j - k \frac{1 + \tau}{2}\,, \end{aligned}$$

for any choice of the branch of the logarithm. The assumption on χ ensures that $b \notin \Gamma$ so that the denominator $\theta(b)$ does not vanish.

The function is unique since the difference of any two is a theta function vanishing at m points. By Corollary A.2, with our assumption on χ, it must vanish identically.

A.3. **Difference equations.** Consider the following problem arising in integrable models. Given $\gamma \in \mathbb{C} - \Gamma$ and functions $A_+(z) \in \Theta_k(\chi_+)$, $A_-(z) \in \Theta_k(\chi_-)$, find $\epsilon(z)$ such that the difference equation

$$A_+(z)Q(z-\gamma) + A_-(z)Q(z+\gamma) = \epsilon(z)Q(z), \tag{20}$$

has a non-trivial solution $Q(z)$ in some $\Theta_m(\chi)$.

A necessary condition is that all terms are theta functions with the same character. So ϵ has to be of level k and $\chi_+(1) = \chi_-(1)$, $\chi_+(\tau)e^{2\pi i\gamma m} = \chi_-(\tau)e^{-2\pi i\gamma m}$ = character of ϵ.

A pair (ϵ, Q) of theta functions (with Q non-trivial) obeying the difference equation (20) will be called an elliptic polynomial solution of (20).

Proposition A.4. *Suppose $A_\pm(z) \in \Theta_k(\chi_\pm)$ with $\chi_+(r+s\tau) = \chi_-(r+s\tau)e^{-4\pi i\gamma ms}$ for all $r+s\tau \in \Gamma$ and some positive integer m. If $a, w_1, \dots, w_m$ are solutions of the system of equations*

$$A_+(w_i)\prod_{j:j\neq i}\theta(w_i - w_j - \gamma) = e^{2a\gamma}A_-(w_i)\prod_{j:j\neq i}\theta(w_i - w_j + \gamma) \qquad i = 1, \dots, m. \tag{21}$$

such that $w_i \neq w_j \mod \Gamma$ for all $i \neq j$, then the functions

$$Q(z) = e^{az}\prod_{j=1}^{m}\theta(z - w_j) \tag{22}$$

and

$$\epsilon(z) = \frac{A_+(z)Q(z-\gamma) + A_-(z)Q(z+\gamma)}{Q(z)} \tag{23}$$

form an elliptic polynomial solution of (20). Conversely, if (ϵ, Q) is an elliptic polynomial solution of (20), then there exists a solution $a, w_1, \dots, w_m$ of the system (21) such that Q is of the form (22) (up to a multiplicative constant) and ϵ is given by (23).

Proof: Let Q be the function defined in (22). The ratio (23) obeys the transformation property of a theta function, but may be singular at the zeros w_i of Q.

The i^{th} equation in the system, or more precisely, the equation equivalent to it if $\gamma \in \mathbb{C} - \Gamma$:

$$A_+(w_i)e^{-\gamma a}\prod_{j=1}^{m}\theta(w_i - w_j - \gamma) + A_-(w_i)e^{\gamma a}\prod_{j=1}^{m}\theta(w_i - w_j + \gamma) = 0,$$

is the condition that the left-hand side of the (20) vanishes at w_i. Thus, if $a, w_1, \dots, w_m$ obey the system of equations, then the quotient ϵ of this left-hand side by $Q(z)$ is regular at w_i and thus everywhere and (ϵ, Q) is an elliptic polynomial solution.

Suppose now that (ϵ, Q) is an elliptic polynomial solution. By Proposition A.1, Q can be written in the form (22) if we normalize it properly. The points w_i are the zeros (modulo Γ) of Q. Then the left-hand side of the difference equation vanishes at w_i so the w_i are a solution of (21) $\square$

Remark. The system of equations is usually written in the form

$$\prod_{j:j\neq i} \frac{\theta(w_i - w_j - \gamma)}{\theta(w_i - w_j + \gamma)} = e^{2a\gamma} \frac{A_-(w_i)}{A_+(w_i)}, \qquad i = 1, \dots, m.$$

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